

# Adaptive Quantization With a One-Word Memory

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We discuss a quantizer which, for every new input sample, adapts its step-size by a factor depending only on the knowledge of which quantizer slot was occupied by the previous signal sample.<sup>1</sup> Specifically, if the outputs of a uniform  $B$ -bit quantizer ( $B > 1$ ) are of the form

$$Y_u = P_u \frac{\Delta_u}{2}; \quad \pm P_u = 1, 3, \dots, 2^B - 1; \quad \Delta_u > 0,$$

the step-size  $\Delta_r$  is given by the previous step-size multiplied by a time-invariant function of the code-word magnitude  $|P_{r-1}|$ :

$$\Delta_r = \Delta_{r-1} \cdot M(|P_{r-1}|).$$

The adaptations are motivated by the assumption that the input signal variance is unknown, so that the quantizer is started off, in general, with a suboptimal step-size  $\Delta_{\text{START}}$ . Multiplier functions that maximize the signal-to-quantization-error ratio (SNR) depend, in general, on  $\Delta_{\text{START}}$  and the input sequence length  $N$ . For example, if the signal is stationary and  $N \rightarrow \infty$ , best multipliers, irrespective of  $\Delta_{\text{START}}$ , have values arbitrarily close to unity. On the other hand, small values of  $N$  and suboptimal values of  $\Delta_{\text{START}}$  necessitate  $M$  values further away from unity. By including an adequate range of values for  $N$  and  $\Delta_{\text{START}}$  in a generalized SNR definition, we show how one can determine stable multiplier functions  $M_{\text{OPT}}$  that are optimal for a given signal.

In computer simulations of 2- and 3-bit quantizers with first-order Gauss-Markovian inputs, we note that, except when the magnitude of the correlation  $C$  between adjacent samples is very high,  $M_{\text{OPT}}$  has the property of calling for fast increases and slow decreases of step-size. We derive optimum multipliers theoretically for two simple cases:

$$M_r^{\text{OPT}} = \left[ \frac{1}{2} + \frac{K^2}{8} P_{r-1}^2 \right]^{\frac{1}{2}} + \delta^2(|P_{r-1}|); \quad C = 0$$

$$M_r^{\text{OPT}} = \frac{|P_{r-1}|}{2^{B-1}} + \delta^2(|P_{r-1}|); \quad C \rightarrow 1.$$

$K$  is a constant depending only on  $B$ , and  $\delta^2$  is a positive correction that is significant only for the last slot:  $|P_{r-1}| = 2^B - 1$ . Using the example of  $C = 0$ , we also show how the approach of specifying  $P_{r-1}$ , explicitly, in the determination of  $\Delta_r$ , is more effective than an earlier procedure<sup>2</sup> where  $\Delta_r$  is determined by past output values  $Y_{r-1}$  (rather than by a function of their components,  $P_{r-1}$  and  $\Delta_{r-1}$ ).

Computer simulations with speech and picture signals have shown, once again, that SNR-maximizing multiplier functions demand step-size increases that are relatively faster than step-size decreases. Values of  $M_{\text{OPT}}$  depend, interestingly, on whether the quantizer is used in a PCM or a DPCM-type coder. In the case of speech signals, we propose corresponding tables of  $M_{\text{OPT}}$  values for  $B = 2, 3, 4$ , and 5. DPCM coding of speech with 3- and 4-bit adaptive quantizers is the subject of a companion paper.<sup>1</sup>

## I. INTRODUCTION

Quantization error, in general, can take one of two distinct forms, overload distortion or granular noise, reflecting, respectively, situations where the quantizer step-size is too small or too large relative to the signal being quantized. This distinction has been widely noted for 1-bit quantizers (delta modulators), and variable step-size quantization has therefore been widely discussed in this context.<sup>3-6</sup> The general idea is to increase the step-size during overload and decrease it during granularity, and to detect those conditions on the basis of observations of the delta modulator bit stream. The step-size adaptations can be either instantaneous<sup>3,5,6</sup> or "syllabic,"<sup>4</sup> and the advantages of adaptation have been shown, among other means, by demonstrations of dynamic range and of SNR gains over nonadaptive quantizers.<sup>6</sup>

The problem of step-size adaptations, as applied to quantizers with more than two output levels, has been less widely studied. It is conventional in such quantizers to take signal nonstationarity into account by means of a suitably designed, time-invariant, nonuniform quantizer.<sup>7</sup> Recently, however, two proposals have incorporated time-variant step-size logics in multibit quantization. The first of these techniques is a syllabically adapting PCM which Wilkinson empirically designed for speech encoding at 10 kb/s.<sup>8</sup> The second proposal is an instantaneously adapting quantizer discussed by Stroh, in the context of differential encoding of Gaussian signals.<sup>2</sup> Syllabic adaption has the advantages that it can be better tailored to a given signal such as

speech and that it can also be designed to provide better resistance to bit errors<sup>4</sup> than instantaneous adaptation. The latter, on the other hand, has the advantages of minimal structure and applicability to different types of signals, and, in relatively noise-protected environments, it constitutes an efficient and simple encoding procedure for signal storage or transmission.

The adaptation that we discuss is instantaneous, and we indicate, at the end of this paper, how it can perform better than Stroh's compandor<sup>2</sup> when working with one word of quantizer (output) memory. We must emphasize here that in each case what is being gained by the adaptation is increased dynamic range rather than an inherent signal-to-noise ratio advantage over a nonadaptive technique. The adaptive techniques presuppose that the input signal variance is unknown. The quantizer step-size cannot therefore be meaningfully preset to an optimized constant value, but must be allowed to adapt itself to signal statistics in a fashion determined by a (time-invariant) adaptation strategy.

The specific quantizer configuration that we consider is characterized by a uniform spacing of nonzero output levels, and Fig. 1 shows a snapshot of the quantizer at sampling instant  $r$  for the example of

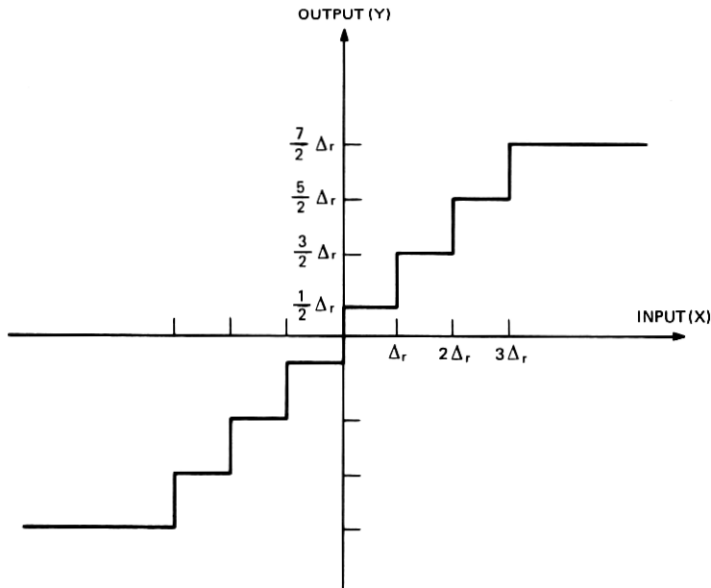


Fig. 1—Uniform quantizer with 8 levels ( $B = 3$ ).

$B = 3$ . The step-size  $\Delta_r$  is adapted, for every new input sample, by a factor depending only on the knowledge of which quantizer slot was occupied by the previous signal sample. More precisely, if the outputs of a  $B$ -bit quantizer ( $B > 1$ ) are of the form

$$Y_u = P_u \frac{\Delta_u}{2}; \quad P_u = \pm 1, 3, \dots, 2^B - 1; \quad \Delta_u > 0, \quad (1)$$

the step-size  $\Delta_r$  is given by the previous step-size multiplied by a time-invariant function of the previous code-word magnitude  $|P_{r-1}|$ :

$$\Delta_r = \Delta_{r-1} \cdot M(|P_{r-1}|). \quad (2)$$

Note that, according to (2), the entire quantizer is "accordioned in" when  $M < 1$  and stretched out when  $M > 1$ . The resulting quantizer is also uniform, with a step-size or "slot width" equal to  $\Delta_r$ . Practical implementations will also include upper and lower limits  $\Delta_{\text{MAX}}$  and  $\Delta_{\text{MIN}}$  for  $\Delta_r$ . This is discussed later in the paper.

The above logic has been recently employed<sup>1</sup> for efficient differential encoding of speech signals at bit rates of 20 to 30 kb/s. The adaptation strategy (2) is indeed arbitrary.\* But it represents, in the manner of the adaptive delta modulator discussed earlier by the author,<sup>6</sup> a very simple, yet nontrivial, type of exponential adaptation, and sets a lower bound on the performance of possible sophistications that may include nonexponential adaptations and the use of longer word memories, i.e., the use of  $P_{r-2}$ ,  $P_{r-3}$ , etc.

An interesting result of this paper is that, for many interesting input signals, the step-size multiplier function  $M(|P|)$  which minimizes the mean-squared quantization error has the interesting property that it demands step-size decreases significantly slower than step-size increases. This is shown to be true for illustrative speech and picture signals and for first-order Gauss-Markovian inputs where the magnitude of the correlation between adjacent signals is not too high (say, less than 0.9).

In Section II, we discuss computer simulations with a first-order Gauss-Markov input. We discuss the simple case of a white signal ( $C = 0$ ) at length. Results show the dependence of signal-to-quantization-error ratio (SNR) on the function  $M(|P|)$  for different values of  $B$  (number of quantizer bits),  $N$  (number of samples in input sequence), and  $\Delta_{\text{START}}$  (initial step-size). We then specify adequate ranges of

\* Schlink has recently described another useful, but perhaps less general, empirical system.<sup>9</sup> Here, the adaptation consists in switching between only two quantizing characteristics.



variation for  $N$  and  $\Delta_{\text{START}}$ , and thence determine a stable multiplier function that is optimal for a white Gaussian signal. Further results include the cases of  $C = 0.5$  and  $0.99$ , and show, for  $B = 2$  and  $3$ , values of  $M_{\text{OPT}}$  and SNR gain over a nonadaptive quantizer. We also provide illustrative histograms of slot occupancies and observed step-sizes and a family of companding curves for a 4-bit quantizer.

In Section III, we derive optimum multipliers theoretically for the examples of  $C = 0$  and  $C \rightarrow 1$ . Results substantiate the values of  $M_{\text{OPT}}$  from the computer simulation. We also compare our technique with that of Stroh<sup>2</sup> and discuss the greater efficacy of our adaptation strategy using the example of  $C = 0$ . Finally, in Section III, we discuss quantizer simulations with speech and picture inputs. We present multiplier functions basically similar to those for Gauss-Markov inputs. Optimal multipliers are found to be slightly different for PCM and DPCM coders. In the case of speech, we provide separate tables of  $M_{\text{OPT}}$  for  $B = 2, 3, 4$ , and  $5$ .

## II. GAUSS-MARKOV INPUTS

Our simulations have employed, as quantizer input, a first-order Gauss-Markovian sequence  $\{X_r\}$  of 10,000 samples generated by the recursive rule

$$X_u = C \cdot X_{u-1} + \sqrt{1 - C^2} \cdot N_u; \quad X_0 = 0, \quad (3)$$

where the samples  $N_u$  are drawn from a zero-mean, unit variance, white Gaussian sequence that is independent of past values of  $\{X_r\}$ . The input sequence generated in (3) is itself Gaussian with a mean of zero, a variance of unity, and a correlation between adjacent samples equal to the preset constant  $C$ .

The quantizer output, by definition, is the output level nearest to the input  $X_r$ . It is formally written as

$$\begin{aligned} Y_r &= \left\{ \left( 2 \left[ \frac{X_r}{\Delta} \right] + 1 \right) \frac{\Delta}{2} \right\} \text{sgn } X_r; & \frac{X_r}{\Delta} < 2^{B-1} \\ &= \left\{ (2^B - 1) \frac{\Delta}{2} \right\} \text{sgn } X_r; & \frac{X_r}{\Delta} \geq 2^{B-1}, \end{aligned} \quad (4)$$

where  $[\cdot]$  stands for "greatest integer in."

The quantization error

$$E_r = Y_r - X_r, \quad (5)$$

has a magnitude that is bounded by  $\Delta/2$  except during overload which is expressed by the second line in eq. (4).

A conventional performance measure is the signal-to-quantization-error ratio

$$\text{SNR} = \frac{\sum X_r^2}{\sum E_r^2}, \quad (6)$$

where summations are assumed to be over the duration of a statistically adequate input sequence.

We also refer in this paper to nonadaptive quantizers for which

$$\begin{aligned} M(|P_{r-1}|) &= 1; & \text{all } P_{r-1} \\ \Delta_r &= \Delta; & \text{all } r, \end{aligned} \quad (7)$$

and the variation of signal-to-quantization-error ratio  $\text{SNR}_{\text{NA}}$  is a function of the constant step-size  $\Delta$  for this case. The step-size which maximizes  $\text{SNR}_{\text{NA}}$  for a nonadaptive quantizer will be referred to as the optimum step-size  $\Delta_{\text{OPT}}$ . Values of  $\Delta_{\text{OPT}}$  and the corresponding values of  $\text{SNR}_{\text{NA}}$ , for different values of  $B$ , have in fact been tabulated by Max<sup>10</sup> for the case of  $C = 0$ . Max's results also specify (via the Gaussian probability density function) the probability  $P_s$  that the  $s$ th slot is occupied in an optimized nonadaptive quantizer:

$$\begin{aligned} P_s &= \text{Prob}(P_u = 2s - 1) + \text{Prob}(-P_u = 2s - 1); \\ s_u &= 1, 2, \dots, 2^{B-1}, \end{aligned} \quad (8)$$

where  $P_u$  is defined by (1). We will see presently that the probability  $P_s$  is also very relevant in the study of an adaptive quantizer when  $C = 0$ .

## 2.1 A General Performance Criterion

Adaptive quantizers are needed, as mentioned earlier, when non-stationary input signals are expected. Our simulations with Gaussian signals utilized a stationary input (3). To make the study of adaptation strategies meaningful in this stationary environment, we shall introduce some unconventional performance measures. For example, consider the ratio

$$\text{SNR}(N, \Delta_{\text{START}}) = \frac{\sum_1^N X_r^2}{\sum_1^N E_r^2}, \quad (9)$$

where summations are over the first  $N$  samples of the input sequence. The dependence of SNR on  $\Delta_{\text{START}}$  is significant only for small values of  $N$ . For large  $N$ , (9) tends to an asymptotic value that is independent of  $\Delta_{\text{START}}$ :

$$\text{SNR}(\infty) \triangleq \lim_{N \rightarrow \infty} \text{SNR}(N, \Delta_{\text{START}}). \quad (10)$$

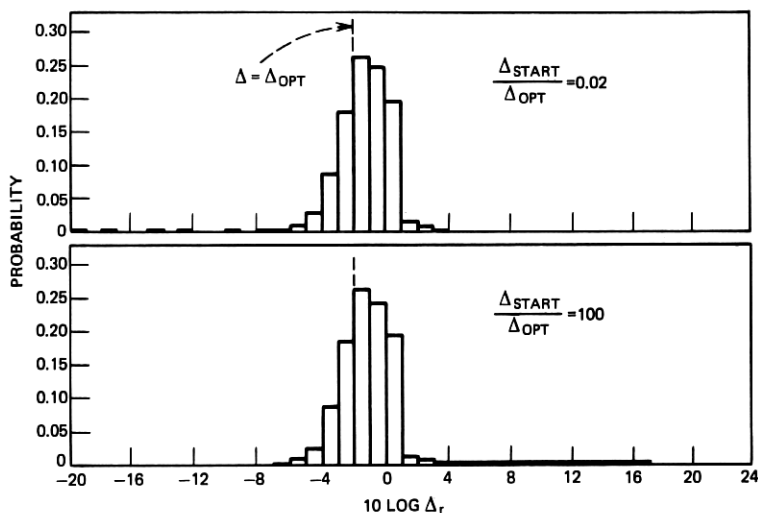


Fig. 2—Step-size histograms ( $B = 3$ ,  $C = 0.5$ ,  $N = 10,000$ ).

In fact, if  $N$  is sufficiently large, the value of  $\Delta_{\text{START}}$  is entirely academic in the study of adaptive quantizers. See the step-size histograms in Fig. 2, for example. Notice how they are independent of  $\Delta_{\text{START}}$ , except for the flat tails representing transient values of  $\Delta$ .

In adaptive quantization, a suitable multiplier function for a given signal should provide a compromise between quickness of response [as measured by the magnitude of (9) for small values of  $N$  and bad values of  $\Delta_{\text{START}}$ ] and satisfactory steady-state performance [as measured by the magnitude of (9) for large values of  $N$  and values of  $\Delta_{\text{START}}$  close to  $\Delta_{\text{OPT}}$ ]. With these opposing factors in mind, we define an average performance index

$$\text{SNR}_{\text{AVE}} \triangleq \frac{1}{20} \sum_N \sum_{\Delta_{\text{START}}} \text{SNR}(N, \Delta_{\text{START}}) \quad (11)$$

for values of  $N = 10, 100, 1000$ , and  $10,000$ , and

$$\Delta_{\text{START}} = \left[ \frac{1}{10}, \frac{1}{\sqrt{10}}, 1, \sqrt{10}, 10 \right] \Delta_{\text{OPT}}.$$

The target values of  $N$  and  $\Delta_{\text{START}}$  above have been chosen with the following factors in mind:

- (i) First, as mentioned earlier, infinitesimally small ranges of values (for example,  $\Delta_{\text{START}} \cong \Delta_{\text{OPT}}$ ; any  $N$ ) are uninteresting

because they can result in  $M_{\text{OPT}}$  values arbitrarily close to the trivial value of unity.

- (ii) On the other hand, overly wide ranges of parameters which include combinations like ( $N = 1$ ,  $\Delta_{\text{START}} = 10^4 \Delta_{\text{OPT}}$ ) reflect pathological situations and lead to multiplier specifications that tend to be quite uncorrelated with the statistical nature of the signal being quantized.
- (iii) As long as the extreme situations in (i) and (ii) are avoided, it has been found that  $M_{\text{OPT}}$  values are not overly sensitive to the actual  $N$  and  $\Delta_{\text{START}}$  values employed in the performance criterion (11), but depend mainly on the statistics of the signal being encoded. In fact, optimal multipliers in this case are merely the best multipliers in a variance-estimating problem (see the theory for  $C = 0$  in Section III) that includes neither  $N$  nor  $\Delta_{\text{START}}$  as a significant parameter.
- (iv) With the aforementioned factors in mind, the specific values of  $N$  and  $\Delta_{\text{START}}$  in (11) were selected to have the following significance for a typical application such as speech quantization. First, the 40-dB range for  $\Delta_{\text{START}}$  reflects an extent of uncertainty (about signal power) which is reasonably characteristic of telephone conversation.<sup>7</sup> Second, when one considers Nyquist-sampled speech for applications like adaptive PCM or adaptive DPCM,<sup>1</sup> the values of  $N$  in (11) correspond at the lower end to about 1 millisecond of speech, and at the higher end to about 1 second of speech. This range clearly includes the range of durations that one may associate with "steady-state" or "stationary" segments in the acoustic waveform. In fact, if one considers phoneme durations, values of  $N$  in the range 100 to 5000 seem to provide an adequate model. It is our contention that by using  $N$  values of this type in an index of performance such as (11), we can very usefully assess  $M$ -functions for quantizing locally stationary signals such as speech, even when simulating the quantizer with a (standard and easily duplicated) stationary Gaussian input. Actually, however, we have carried out completely independent simulations with real speech signals as well (Section IV), and the results of this section are directed toward the quantization of Gaussian inputs as such.

## 2.2 Multiplier Functions for $B = 2$ , $C = 0$

Table I illustrates the nature of the SNR function (9) for two multiplier functions in a 2-bit quantizer. The first multiplier function

TABLE I—EXAMPLE OF SNR FUNCTIONS FOR  $B = 2$ ,  $C = 0$   
(ENTRIES IN dB)

$20 \log \left( \frac{\Delta_{\text{START}}}{\Delta_{\text{OPT}}} \right)$	Values of $N$			
	10	100	1000	10,000
	$M_1 = 0.8$		$M_2 = 1.6$	
-20	6.4	7.2	7.4	7.3
-10	10.5	8.9	7.9	7.3
0	9.7	8.3	7.6	7.3
10	5.8	7.2	7.4	7.2
20	-5.9	4.2	7.1	7.3
	$M_1 = 0.98$		$M_2 = 1.04$	
	10	100	1000	10,000
	$M_1 = 0.8$		$M_2 = 1.6$	
-20	1.6	3.8	8.4	9.1
-10	5.2	5.8	8.9	9.2
0	10.7	8.0	9.4	9.2
10	0.0	5.9	9.0	9.2
20	-13.2	-5.0	3.5	8.1

shows quicker response (better SNR values for  $N = 10$  and  $100$ ), while the second function achieves a better asymptotic value of SNR (at  $N = 10,000$ ). Obviously, the poor asymptotic performance of the first  $M$ -function is due to overly abrupt step-size oscillations in the "steady-state," while the inferior performance of the second  $M$ -function for small  $N$  is due to sluggish adaptations of  $\Delta$  when  $\Delta_{\text{START}}$  is suboptimal.

Table II compares several  $M$ -functions\* for a 2-bit quantizer on the basis of (11). The functions included represent a subset of many more functions which were simulated and compared on the basis of  $\text{SNR}_{\text{AVE}}$ . The best value of 6.8 dB has been noted for  $M_1 = 0.80$ ,  $M_2 = 1.60$ , although this function provides a clearly nonmaximal asymptotic performance (Table I). The first five functions in Table II also satisfy

TABLE II—COMPARISON OF MULTIPLIER FUNCTIONS ( $B = 2$ ,  $C = 0$ )

$M_1$	$M_2$	$\text{SNR}_{\text{AVE}}$ (dB)
0.71	2.00	5.9
0.80	1.60	6.8
0.90	1.20	6.5
0.95	1.10	6.1
0.98	1.04	5.3
0.95	1.20	5.9
0.50	2.00	5.8
0.90	1.10	5.2

\* Whenever there is no scope for confusion, we shall use the symbols  $M_1$ ,  $M_2$ ,  $M_3$ , and  $M_4$  instead of  $M_r(1)$ ,  $M_r(3)$ ,  $M_r(5)$ , and  $M_r(7)$ .

the interesting constraint suggested by Goodman:<sup>11</sup>

$$M_1^2 \cdot M_2 \cong M_1^{0.67} M_2^{0.33} \cong M_1^{P_1} \cdot M_2^{P_2} \cong 1, \quad (12)$$

where  $P_1 \cong 0.67$  and  $P_2 \cong 0.33$  are the probabilities of inner- and outer-slot occupancy in a nonadaptive quantizer with an optimal  $\Delta_{\text{OPT}}$  for the Gaussian input. Goodman conjectures that the probabilities of using  $M_1$  and  $M_2$  in a well-designed adaptive quantizer should indeed be equal to the parameters  $P_1$  and  $P_2$  of the nonadaptive quantizer. A constraint of the form (12) then represents a stability

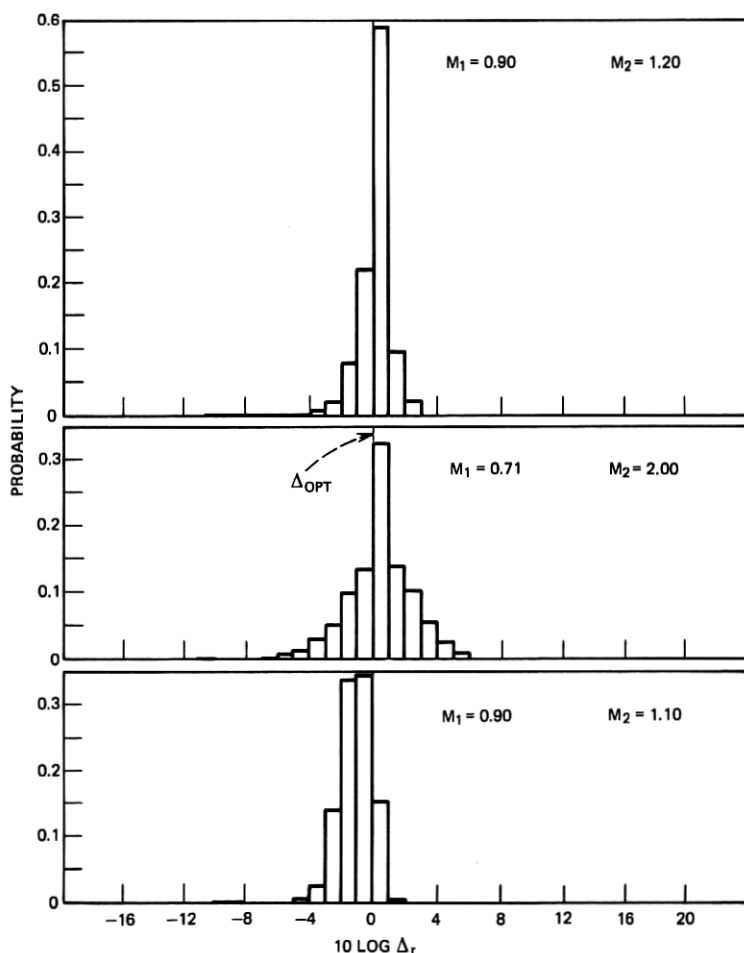


Fig. 3—Step-size histograms ( $B = 2$ ,  $C = 0$ ,  $N = 10,000$ ).

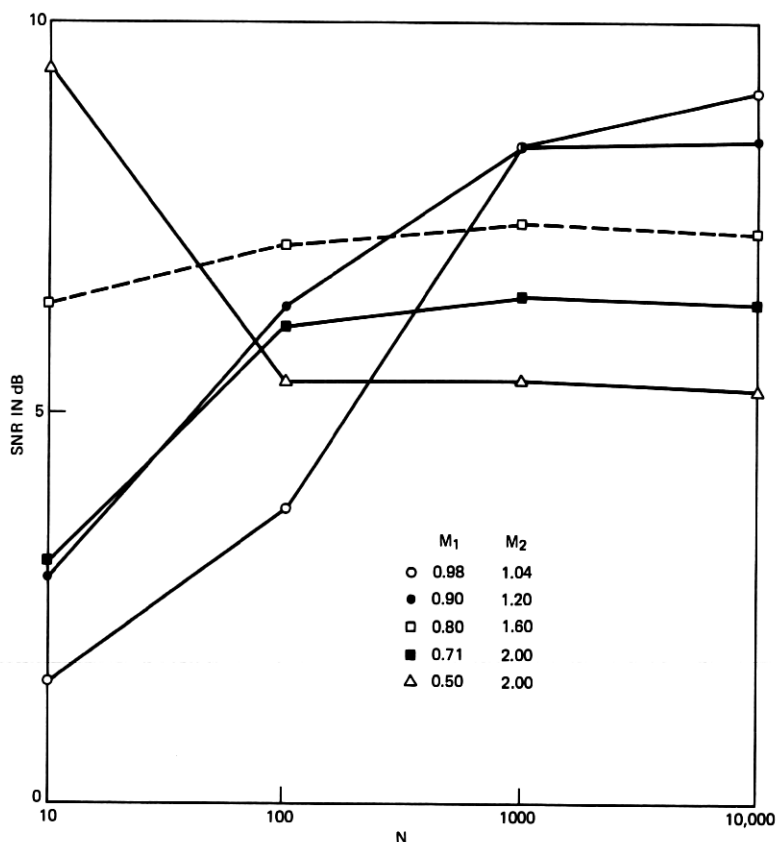


Fig. 4—Comparison of multiplier functions ( $B = 2$ ,  $C = 0$ ,  $\Delta_{\text{START}} = 0.1$   $\Delta_{\text{OPT}}$ ).

criterion which specifies that the random step-size  $\Delta_r$  neither grows out of bounds, independently of the input, nor decays to infinitesimal values. This criterion has been discussed earlier in the context of adaptive delta modulation with a 1-bit memory.<sup>6</sup>

The desirability of constraint (12) on step-size multipliers is also demonstrated by the step-size histograms in Fig. 3. The multiplier pairs (0.9, 1.2) and (0.71, 2.0) satisfy constraint (12), and the corresponding histograms have the desirable property that they are centered on  $\Delta_{\text{OPT}}$  although they have different dispersions (suggesting differences in quickness of response and steady-state performance). The function (0.9, 1.10), on the other hand, produces a histogram whose mode is clearly displaced from  $\Delta_{\text{OPT}}$ . This suggests that (0.9, 1.10) falls in a

TABLE III—SNR FUNCTION FOR  $M_1 = 0.90$ ,  $M_2 = 0.90$ ,  $M_3 = 1.25$ ,  $M_4 = 1.75$  ( $B = 3$ ,  $C = 0$ , ENTRIES IN dB)

$20 \log \left( \frac{\Delta_{\text{START}}}{\Delta_{\text{OPT}}} \right)$	Values of $N$			
	10	100	1000	10,000
-20	11.9	11.2	12.7	12.7
-10	14.5	11.8	12.6	12.7
0	15.7	11.2	12.9	12.7
+10	11.8	11.6	12.5	12.7
+20	-1.6	8.7	12.3	12.7

class of inefficient multiplier functions; this is attributed to the fact that the function (0.9, 1.10) clearly violates requirement (12) above.<sup>11</sup>

Finally, in Fig. 4, we show SNR (6) as a function of  $N$  for a fixed value of  $\Delta_{\text{START}}$ , and for different  $M$ -functions. It is once again apparent that the adaptation function (0.8, 1.6) provides an attractive combination of responsiveness and asymptotic performance for  $B = 2$ .

### 2.3 Multiplier Functions for $B = 3$ , $C = 0$

Table III demonstrates the nature of the SNR function (9) for  $B = 3$  and a specific multiplier function. Table IV uses the performance criterion (11) to show the efficiency of this multiplier function (0.9, 0.9, 1.25, 1.75). As in the 2-bit example, the  $M$ -functions in Table IV are only a subset of a much larger set of  $M$ -functions which were simulated and compared on the basis of  $\text{SNR}_{\text{AVE}}$ . We have only included the most interesting functions from our search for maximum  $\text{SNR}_{\text{AVE}}$ . The first three  $M$ -functions in Table IV satisfy a stability constraint analogous to (11):

$$M_1^{0.46} M_2^{0.31} M_3^{0.16} M_4^{0.07} = M_1^{P_1} \cdot M_2^{P_2} \cdot M_3^{P_3} \cdot M_4^{P_4} = 1. \quad (13)$$

It is interesting that the best function in Table IV belongs to the class of functions obeying (13). Notice also that the reduction of the number of distinct step-size multipliers (second row in Table IV) leads to a

TABLE IV—COMPARISON OF MULTIPLIER FUNCTIONS ( $B = 3$ ,  $C = 0$ )

$M_1$	$M_2$	$M_3$	$M_4$	$\text{SNR}_{\text{AVE}}$ (dB)
0.90	0.90	1.25	1.75	11.7
0.90	1.00	1.00	1.75	11.4
0.5	1.0	1.0	2.0	9.6
0.3	0.9	1.5	2.1	8.9



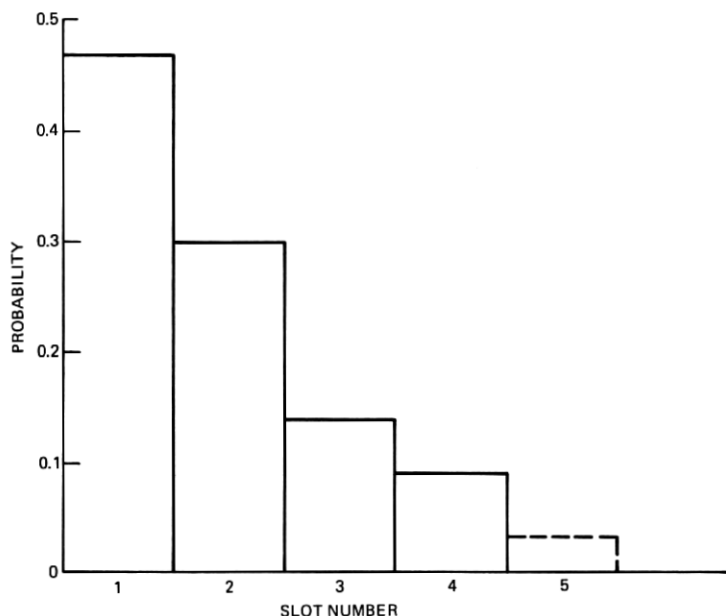


Fig. 5—Histogram of slot occupancies ( $B = 3$ ,  $C = 0$ ,  $N = 10,000$ ).

marginal decrease of  $\text{SNR}_{\text{AVE}}$ . This tolerance to a reduction of the number of distinct multipliers does seem to extend, although in lesser measure, to larger values of  $B$  and to speech and picture signals.

Finally, Fig. 5 shows a histogram of slot occupancies for the best  $M$ -function in Table IV. The number of quantizer slots or output levels is equal to four (neglecting signs), and the dotted fifth slot refers to the overload probability that has been accumulated into the fourth bar of the histogram. It is interesting that, despite step-size adaptations, the Gaussian nature of the input density function shows up in the histogram. The heights of the bars in Fig. 5 represent experimental slot probabilities of 0.47, 0.30, 0.14, and 0.09. Notice again that, in the manner of (13):

$$0.9^{0.47} \cdot 0.9^{0.30} \cdot 1.25^{0.14} \cdot 1.75^{0.09} = 0.994 \cong 1. \quad (14)$$

#### 2.4 Comparison of Adaptive and Nonadaptive Quantizers

Table V summarizes the nature of optimal multiplier functions for  $B = 2$  and 3. These functions are obtained on the basis of criterion (11). Values of  $M$  are generally rounded, representing broad optima,

TABLE V—QUANTIZATION OF GAUSS-MARKOV INPUTS [ENTRIES ARE SNR (10,000,  $\Delta_{OPT}$ ) VALUES IN dB]

$B$	$C$	0.00	0.50	0.99
2	$SNR_{NA}$	9	9	9
	$SNR_A$	7	8	11
	$M(1)$	0.8	0.8	0.5
	$M(2)$	1.6	1.6	2.0
3	$SNR_{NA}$	14	14	14
	$SNR_A$	13	13	16
	$M(1)$	0.90	0.90	0.30
	$M(2)$	0.90	0.90	0.90
	$M(3)$	1.25	1.25	1.50
	$M(4)$	1.75	1.75	2.10

and the precision in the specification of  $M$  values may be as bad as  $\pm 5$  percent in some cases.

To provide a fair comparison with optimal nonadaptive quantizers, the performance figure used in Table V is the asymptotic value (10). Formally, the notation used in the table is as follows:

$$SNR = SNR(10,000, \Delta_{OPT}). \quad (15)$$

The subscript A refers to the adaptive quantizer with step-size multipliers optimized using (11), while the subscript NA refers to a non-

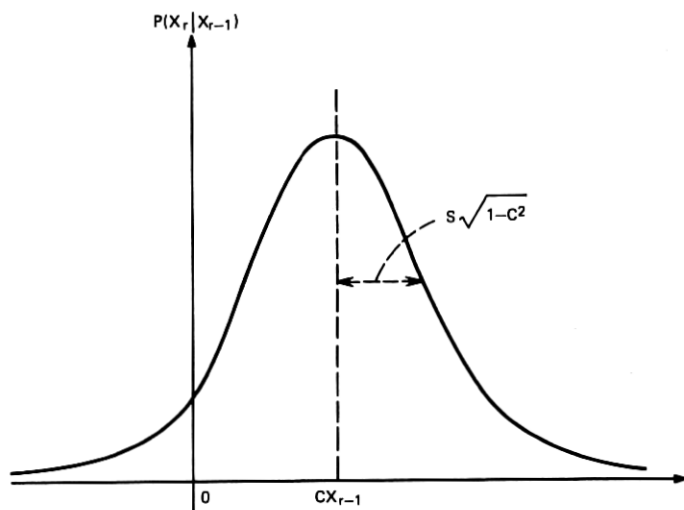


Fig. 6—Conditional density function of quantizer input.

adaptive quantizer with constant step-size  $\Delta_{\text{OPT}}$ . The SNR values are in dB, and are rounded to the nearest integer.

Note that negative values of  $C$  are not included in the table. The assumption of a symmetrical quantizer (Fig. 1) renders the quantizer design independent of the sign of  $C$ . Specifically, the quantizer input  $X_r$  has a probability density function (conditioned to  $X_{r-1}$ ) that is sketched in Fig. 6; the optimum step-size is that which fits the quantizer to this density function in a way that minimizes the sum of overload error variance and granular error power. This optimum depends only on the disposition of the PDF in Fig. 6 and the magnitude of the nonzero mean, not the sign of it.

Finally, Table V assumes that no constraints exist on the minimum and maximum values of step-size. Practical implementations will, of course, involve such constraints (see Fig. 8), as well as constraints on actual multiplier values. Significant conclusions from Table V are the following:

(i) Except for  $C = 0.99$ , optimal multipliers are such that step-size decreases are always slower than step-size increases. The observation has been found to extend for  $B = 4$  also and, as seen later (Section IV), to the quantization of speech and picture signals as well.

The need for fast increases of step-size and slow decreases thereof may be physically explained as follows. Quantization errors during overload tend to be more harmful than those during granularity, in that the magnitude of granular error is restricted, by definition, to a half step-size, while no such simple constraint exists for an overload error. It is therefore reasonable to decrease step-sizes (relatively) slowly to avoid unduly small step-sizes leading to the harmful overload errors. The observation is obviously less significant for a coarser quantizer than for a finer quantizer because granular errors in the former are more comparable in magnitude to overload errors and hence more equally harmful. This is indeed reflected in Table V. Note that, for a given value of  $C$ , the disparity in rates of step-size increases and step-size decreases is least for the coarser quantizer ( $B = 2$ ).

There is an alternative explanation for (i) above, which also clarifies why the disparity between the speeds of step-size increase and step-size decrease is less apparent for large values of  $C$ . Refer to the stability constraints (12) and (13), as discussed for the case of  $C = 0$ . It turns out that in the uniform (nonadaptive) quantization of a Gaussian signal, the probability  $P_s$  (8) is a monotonically decreasing function of  $s$ . It follows then, as seen in (12) and (13), that multipliers for step-size decreases have greater probabilities of being employed, and hence

must lead to slower step-size changes each time they are actually used. Explicitly, for  $B = 2$ , (12) can be rewritten:

$$\frac{\ln M_2}{\ln(1/M_1)} = \frac{P_1}{P_2}. \quad (16)$$

Obviously, then, if  $P_1 > P_2$ , the step-size increase (as given by  $M_2$ ) is faster than the step-size decrease (as given by  $M_1$ ).

The argument for nonzero values of  $C$  is very similar, except that the probabilities  $P_s$  peculiar to a Gaussian probability density function should now be replaced by probabilities  $P_s(C)$  that refer to the uniform quantization of the asymmetrical conditional PDF in Fig. 6. Apparently, the probabilities  $P_s(C)$  are not monotonically decreasing for  $C = 0.99$ . This is why the requirement of relatively more rapid step-

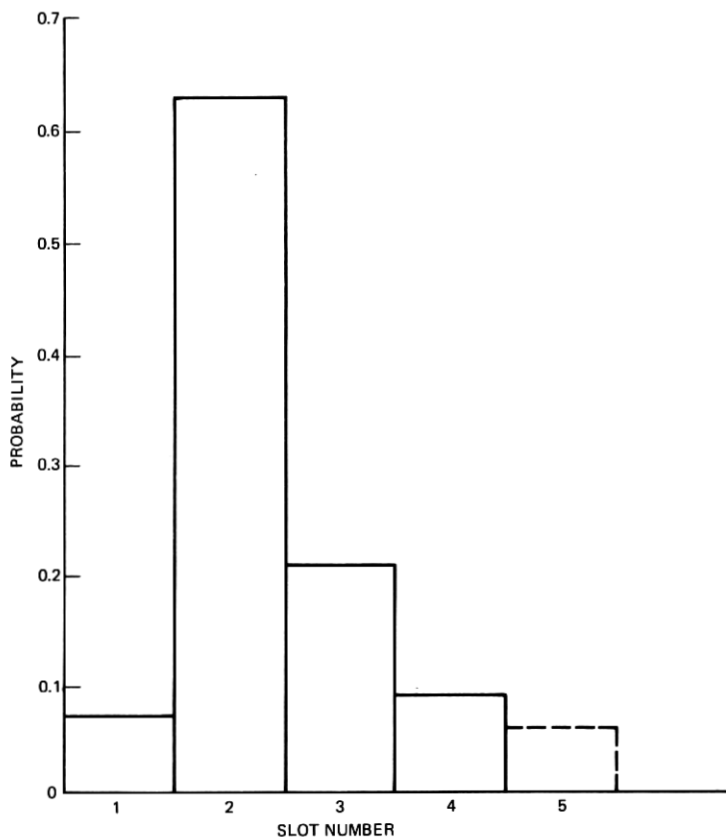


Fig. 7—Histogram of slot occupancies ( $B = 3$ ,  $C = 0.99$ ,  $N = 10,000$ ).

size increases is waived for the example of  $C = 0.99$  (while being still true for  $C = 0.5$ ).

Figure 7 shows a histogram of slot-occupancies for  $B = 3$  and  $C = 0.99$ . Compare this non-monotonic PDF with the Gaussian histogram for  $C = 0$  (Fig. 5). In analogy with (13), the stability criterion associated with Fig. 7 is

$$0.3^{0.07} 0.9^{0.63} 1.5^{0.21} 2.1^{0.09} = 0.993 \cong 1. \quad (17)$$

Finally, it should be mentioned that the (relatively) slow step-size decreases in Table V are fast enough, in an absolute sense, for typical quantizer applications. For example, if  $M = 0.95$ , and a step-size decrease of 20 dB is needed for adaptation to an idle-channel situation in speech quantization, the time needed for such adaptation will be 45 samples. For Nyquist-sampled speech, this is only about 5 ms.

(ii) Although the quantization problem for  $C = 0.5$  is qualitatively similar to that for  $C = 0.99$  (Fig. 6), we note that results for  $C = 0.5$  (Table V) are nearly identical with those for  $C = 0$ . The differences in  $M_{\text{OPT}}$  values that are caused by a nonzero  $C = 0.5$  were apparently too small to be detected in our finite search for best multipliers.

(iii) Referring again to Table V, the best adaptive quantizers seem to have an SNR advantage over the nonadaptive scheme (working with an optimal step-size) only for very highly correlated inputs. In fact, in many instances, the SNR gain resulting from adaptation is seen to be negative (due, evidently, to overly abrupt manipulations of step-size).

The reason for using an adaptive quantizer in these situations is only to facilitate quantizations with much less knowledge of the input—equivalently, with much less knowledge of  $\Delta_{\text{OPT}}$  than is necessary for an equivalent performance in the nonadaptive case. In other words, step-size adaptations increase the dynamic range of the quantizer and enable it to handle inputs with large amplitude variations, such as nonstationary signals.

The above idea has already been demonstrated by the asymptotic SNR values in Tables I and III. To provide a more application-oriented illustration, we undertook two extensions of our computer simulation. These experiments employed  $B = 4$ ,  $C = 0.5$ , and the following multiplier function:

$$(0.90, 0.90, 0.95, 1.0, 1.2, 1.5, 1.8, 2.1). \quad (18)$$

Finite step-size dictionaries were used, determined by maximum and minimum step-sizes  $\Delta_{\text{MAX}}$  and  $\Delta_{\text{MIN}}$ . The starting step-size was set

equal to  $\Delta_{OPT}$ , subject, however, to modification because of the constraints  $\Delta_{MAX}$  and  $\Delta_{MIN}$ .

In the first of these extensions, the step-size dictionary had the characterization

$$\Delta_{MAX} \cdot \Delta_{MIN} = \Delta_{OPT}^2 \quad (19)$$

$$\Delta_{MAX}/\Delta_{MIN} = R \quad (20)$$

and the quantizer performance was studied in terms of SNR (10,000,  $\Delta_{OPT}$ ) as a function of  $R$ . It was reassuring to note that the SNR was constant to within 1 dB for sample values of  $R$  in the range 1 to  $\infty$ —due, no doubt, to the safe design feature (19). In fact, a maximum SNR was noted for a noninfinite value of  $R$ .

In a more revealing second experiment, the quantizer was “centered” at a value  $\Delta_{MID}$  not necessarily equal to  $\Delta_{OPT}$ :

$$\Delta_{MAX}\Delta_{MIN} = \Delta_{MID}^2 \quad (21)$$

and the performance was measured as a function of  $\Delta_{MID}$  for values of  $R$  (20) equal to 1, 10, and 100. Note that  $R = 1$  refers to the non-adaptive case.

Figure 8 plots these results. The monotonic improvement of dynamic

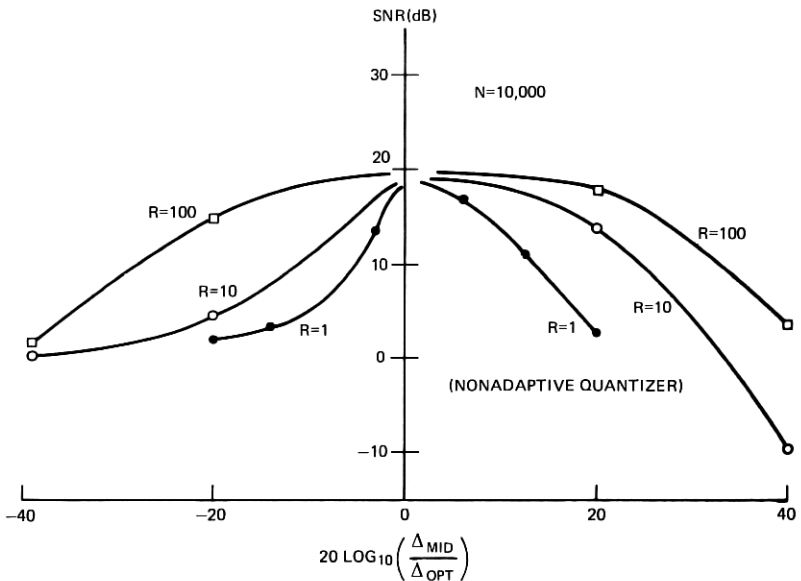


Fig. 8—Companding characteristics ( $B = 4$ ,  $C = 0.5$ ).

range with increasing  $R$  is apparent. It is expected<sup>1</sup> that practical quantizers can be designed with values of  $R$  equal to 100 or more.

### III. THEORETICAL DERIVATION OF OPTIMAL MULTIPLIERS

In this section, we shall regard the adaptive quantization problem as one of learning signal variance. In other words, the problem of determining an optimum instantaneous step-size  $\Delta_r$  is regarded as being tantamount to that of finding the best estimate at time  $r$  of the conditional standard deviation  $S_r$  of the quantizer input; and of setting  $\Delta_r$  proportional to this estimate:

$$\Delta_r^{\text{OPT}} = K(B) \cdot \hat{S}_r(\Delta_{r-1}, P_{r-1}). \quad (22)$$

#### 3.1 Case of $C = 0$

The constant  $K$  is an obvious function of the number of quantizer levels and, hence, of  $B$ . For the problem of uniform quantization of a zero-mean Gaussian signal, Max's Table II<sup>10</sup> specifies the following values for  $K(B)$ .\*

$$\begin{aligned} K(1) &= 1.596, & K(2) &= 0.996, \\ K(3) &= 0.586, & K(4) &= 0.335. \end{aligned} \quad (23)$$

The dependence of  $\hat{S}_r$  on  $\Delta_{r-1}$  and  $P_{r-1}$  (22) is, of course, characteristic of an adaptation strategy which uses a 1-word memory.

We now propose that the variance of  $X_r$  be estimated as the average of the squares of (i)  $X_{r-1}$ , the most recent quantizer input, and (ii)  $\hat{S}_{r-1}$ , the most recent estimate of  $S$ . In other words, let

$$\hat{S}_r^2 = \frac{1}{2}(X_{r-1}^2 + \hat{S}_{r-1}^2).^\dagger \quad (24)$$

We next recall the identity

$$X_{r-1} = Y_{r-1} - E_{r-1} = \frac{P_{r-1}\Delta_{r-1}}{2} - E_{r-1}, \quad (25)$$

where  $E_{r-1}$  is the quantization error. Furthermore, by virtue of the basic algorithm (22), we suggest that

$$\hat{S}_{r-1}^2 = \Delta_{r-1}^2 / K^2 \quad (26)$$

Let us use (25) and (26) in (24) and set the resulting value of  $\hat{S}_r$  in (22). We obtain, after some algebra:

$$(\Delta_r^{\text{OPT}} / \Delta_{r-1})^2 = \frac{K^2}{2} \left[ \frac{P_{r-1}^2}{4} + \frac{1}{K^2} + \frac{1}{\Delta_{r-1}^2} (E_{r-1}^2 - E_{r-1}\Delta_{r-1}P_{r-1}) \right]. \quad (27)$$

\* These  $K$  values are relevant for  $C = 0$  because, in this case, the conditional density function (Fig. 6) is indeed zero-mean Gaussian.

<sup>†</sup> In general, one may consider a weighted average of the type  $ux^2 + vs^2$ . The case of  $u = 0$  will be appropriate for "steady-state" operation, and the use of  $v = 0$  will be appropriate for a "transient" situation. The need for time-invariant step-size multipliers suggests a compromise design characterized by a weighting of the type  $u = v = 0.5$ .

$E_{r-1}$  is an unknown random variable, but the following can be said about its role in (27):

First, the  $E_{r-1}^2$  term is significant only for the last quantizer slot in which, due to possible overload,  $E_{r-1}^2$  can be arbitrarily large. Furthermore, for this end slot the  $-E_{r-1}\Delta_{r-1}P_{r-1}$  term tends to be positive. Notice, from definition (25),  $E$  is negative in overload when  $P$  is positive and vice versa.

For the remaining quantizer slots,  $E_{r-1}^2$  is again positive but no longer significant, and  $-E_{r-1}\Delta_{r-1}P_{r-1}$  is expected to be negligible as well, on the average. This is by virtue of the uniform PDF approximation for granular errors

$$P(E) = 1/\Delta; \quad -\frac{\Delta}{2} < E_{\text{gran}} < \frac{\Delta}{2} \quad (28)$$

and a consequent decorrelation of output  $P\Delta$  and error  $E$ .

The optimum multiplier function [square root of (27)] can therefore be expressed in the form

$$M_r^{\text{OPT}} = \left[ \frac{1}{2} + \frac{K^2}{8} P_{r-1}^2 \right]^{\frac{1}{2}} + \delta^2(|P_{r-1}|); \quad C = 0; \quad (29)$$

where  $\delta^2$  is a positive correction term that is significant only for the end slot:

$$\delta^2(|P_{r-1}|) \cong 0 \quad \text{if} \quad |P_{r-1}| \neq 2^B - 1. \quad (30)$$

Table VI compares the  $M$  values from (29) with those from the simulation in Section II.

### 3.2 Comparison With Stroh's Adaptation Logic ( $C = 0$ )

Consider, in place of (24), a simpler variance estimation of the type considered by Stroh:<sup>2</sup>

$$\hat{S}_r^2 = X_{r-1}^2. \quad (31)$$

This results in, by virtue of (25), (22), and arguments similar to those at the end of the previous paragraph, a multiplier function of the form

$$M_r = \frac{K}{2} |P_{r-1}| + \delta^2(|P_{r-1}|); \quad (32)$$

$$\delta^2(|P_{r-1}|) \cong 0 \quad \text{if} \quad |P_{r-1}| \neq 2^B - 1.$$

Table VI lists values of  $M_r^{\text{OPT}}$  (29),  $M_r$  (32), and the experimental optima  $M_{\text{EXP}}$  from Table V. Values of  $K$  have been taken from (23).

Notice how  $M_r^{\text{OPT}}$  provides a better specification of optimal multi-



TABLE VI—COMPARISON OF MULTIPLIER FUNCTIONS

$B = 2$			$ P_{r-1} $	$B = 3$		
$M_r$	$M_r^{\text{OPT}}$	$M_{\text{EXP}}$		$M_r$	$M_r^{\text{OPT}}$	$M_{\text{EXP}}$
0.50	0.79	0.80	1	0.29	0.75	0.90
$1.50 + \delta^2$	$1.27 + \delta^2$	1.60	2	0.87	0.94	0.90
—	—	—	3	1.45	1.26	1.25
—	—	—	4	$2.00 + \delta^2$	$1.61 + \delta^2$	1.75

pliers than does  $M_r$ . Furthermore, as  $B$  increases, the constant  $K$  approaches zero and the theoretical multiplier functions for the innermost slot ( $P_{r-1} = \pm 1$ ) have the following limiting behavior:

$$\lim_{B \rightarrow \infty} M_r(1) = 0 \quad (33)$$

$$\lim_{B \rightarrow \infty} M_r^{\text{OPT}}(1) = \sqrt{1/2} = 0.71. \quad (34)$$

Simulations with  $B = 4$  and  $5$  have verified that the trend in (34) is indeed more realistic than that in (33).

It should be mentioned that the adaptation strategy (31) is only the simplest case of Stroh's<sup>2</sup> method which has a general variance estimator of the form

$$\hat{S}_{r,n}^2 = \frac{1}{n} \sum_{u=1}^n X_{r-u}^2. \quad (35)$$

It is interesting, nevertheless, that for the same length ( $n = 1$  or one-word) of quantizer memory, our adaptation rule specifies better step-size multipliers, as seen in Table V. In fact, the use of  $M_r^{\text{OPT}}$  yields for ( $B = 3$ ,  $C = 0$ ), an SNR (for Gaussian signals) which is better than what Stroh reports for  $n = 2$  (10 dB;  $N = 2500$ ) in his Fig. 3.3. With the experimentally optimized  $M$ -function (Table V), we indeed do significantly better and the SNR value of 12.7 dB for this case is equivalent to  $n = 6$  in Stroh's logic and falls short of the optimum ( $n = \infty$  in Stroh) by not much more than 1 dB.

The efficiency of our logic is clearly attributable to the way we exploit quantizer memory, namely, in terms of  $P$  and  $\Delta$ , rather than in terms of the product of the two quantities (the quantizer output  $Y$  used by Stroh). Physically, the use of  $P\Delta$  for adaptation seems to wipe out some of the "overload" and "underload" cues that an individual knowledge of  $P$  and  $\Delta$  preserves.

TABLE VII—COMPARISON OF THEORETICAL ( $M_r^{\text{OPT}}$ ) AND EXPERIMENTAL ( $M_r^{\text{EXP}}$ , IN PARENTHESES) MULTIPLIERS

$B$	$C$	0	0.99
2	$M(1)$	0.79(0.8)	0.5(0.5)
	$M(2)$	$[1.27 + \delta^2](1.6)$	$[1.5 + \delta^2](2.0)$
3	$M(1)$	0.75(0.9)	0.25(0.3)
	$M(2)$	0.94(0.9)	0.75(0.9)
	$M(3)$	1.26(1.25)	1.25(1.5)
	$M(4)$	$[1.61 + \delta^2](1.75)$	$[1.75 + \delta^2](2.1)$

3.3 Case of  $C \rightarrow 1$ 

When the adjacent signal correlation  $C$  approaches unity, the conditional PDF (probability density function) of  $X_r$  approaches a Gaussian spike centered at  $CX_{r-1}$  (Fig. 6). The width of the spike is proportional to the square root of  $(1 - C^2)$ , and therefore approaches zero irrespective of the value of signal variance  $S$ . The adaptive quantization problem is no longer one of variance estimation. It will consist, instead, in a "fool-proof" strategy of the following type: Select a step-size  $\Delta_r$  such that the PDF spike at  $CX_{r-1}^*$  falls right in the middle of the positive (or negative) half of the quantizer range, assuming that  $CX_{r-1}$  is positive (or negative). If we recall that a  $B$ -bit quantizer has a half-range width equal to  $2^{B-1}\Delta$ , we see the requirement (assuming positive quantities throughout) is:

$$CX_{r-1} = \frac{2^{B-1}\Delta_r}{2}; \quad C \rightarrow 1. \quad (36)$$

The logic clearly provides simultaneous protection against both overload and underload. Utilizing the estimate of  $X_{r-1}$  (25) in (36), we obtain the condition

$$C \left[ \frac{P_{r-1}\Delta_{r-1}}{2} - E_{r-1} \right] = 2^{B-2}\Delta_r; \quad C \rightarrow 1. \quad (37)$$

Equivalently, with usual assumptions on the quantization error  $E_{r-1}$ ,

$$\lim_{C \rightarrow 1} M_r^{\text{OPT}} = \frac{\Delta_r}{\Delta_{r-1}} = \frac{|P_{r-1}|}{2^{B-1}} + \delta^2(|P_{r-1}|) \quad (38)$$

$$\delta^2(|P_{r-1}|) \cong 0 \quad \text{if } P_{r-1} \neq 2^B - 1.$$

\* See the spike in the histogram of Fig. 7.

### 3.4 Comparison with Simulation Results

Results for a general value of  $C$  ( $0 < C < 1$ ) can in principle be attempted on the basis of a general PDF such as Fig. 6. However, tractable derivations seem to require too many simplifying assumptions to make the theory worthwhile, especially in view of the observation (Table V) that the correlation becomes significant only if  $C \rightarrow 1$ . We therefore conclude this section by merely listing, in Table VII, theoretical step-size multipliers for  $C = 0$  and 0.99 [from (29) and (38)] together with the experimentally optimized multipliers from Section II.

## IV. QUANTIZER SIMULATIONS WITH SPEECH AND PICTURE SIGNALS

In this section, we present results from computer simulations of the adaptive quantizer with speech and picture inputs.

The results in Table VIII refer to a low-pass-filtered speech signal (about a second long), and a single frame of picture input (the face of Karen in *Picturephone*<sup>®</sup> format<sup>6</sup>). Listed are step-size multipliers found, by search procedure, to maximize an asymptotic SNR (10) as measured over the entire length ( $N \gg 10,000$ ) of the input sequences. The following observations are of interest:

- (i) The signal PDF seems to have a significant effect (presumably through overload statistics and the end-slot correction  $\delta^2(|P_{r-1}|)$  of Section III) on the largest step-size multiplier. Note the value of  $M_4$  for picture input.

TABLE VIII—STEP-SIZE MULTIPLIERS FOR ILLUSTRATIVE SPEECH AND PICTURE SIGNALS (ENTRIES IN PARENTHESES REFER TO PICTURES)

$B \backslash$ Coder Type	PCM	DPCM
2	0.6, 2.2	0.8, 1.6
3	0.85, 1, 1, 1.5 (0.9, 0.95, 1.5, 2.5)	0.9, 0.9, 1.25, 1.75 (0.9, 0.95, 1.5, 2.75)
4	0.8, 0.8, 0.8, 0.8, 1.2, 1.6, 2.0, 2.4	0.9, 0.9, 0.9, 0.9, 1.2, 1.6, 2.0, 2.4
5	0.85, 0.85, 0.85, 0.85, 0.85, 0.85, 0.85, 0.85, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6	0.9, 0.9, 0.9, 0.9, 0.95, 0.95, 0.95, 0.95, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3

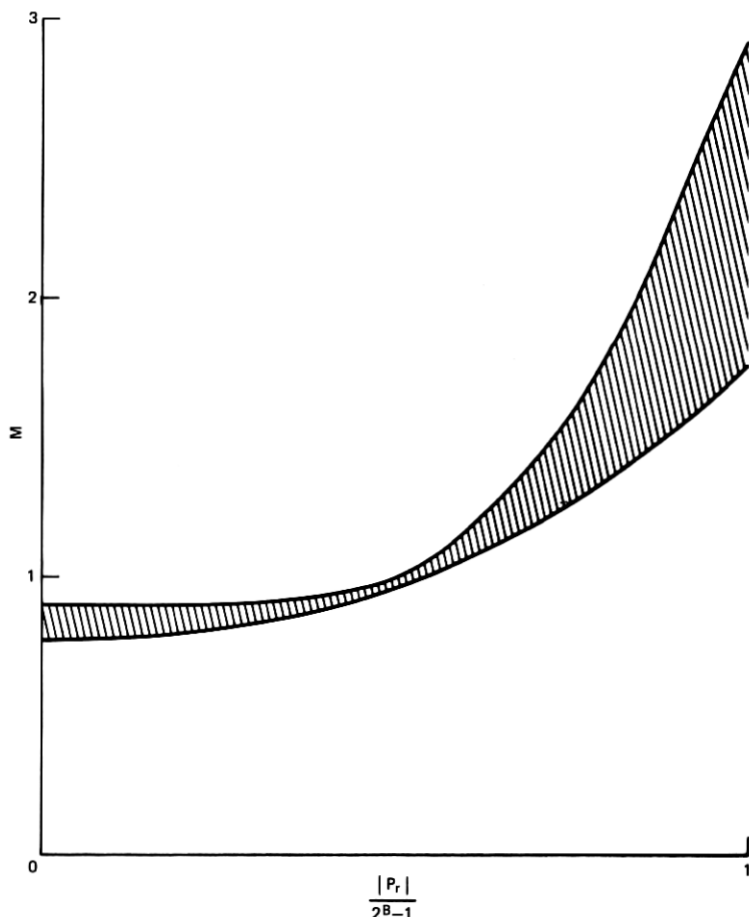


Fig. 9—Desirable form of the multiplier function  $M$  for the adaptive quantization of speech signals ( $B > 2$ ) and first-order Gauss-Markov signals which are not highly correlated (say,  $C \gtrsim 0.5$ ).

- (ii) Differentiation has the effect of decreasing adjacent sample correlation. This seems to explain differences in multipliers as applied to PCM and differential PCM quantizers for speech. Note that the effect is most pronounced for  $B = 2$ .
- (iii) Although the input signals are not first-order Markovian, the multipliers have the earlier-mentioned property that step-size increases are relatively more rapid than step-size decreases. Refer to the general diagram in Fig. 9.

TABLE IX—COMPARISON OF SPEECH QUANTIZERS  
(ENTRIES ARE SNR VALUES IN dB)

$B$	Logarithmic PCM with $\mu$ -law Quantization	Adaptive PCM with Uniform Quantization	Adaptive DPCM with Uniform Quantization	Adaptive DPCM with Nonuniform Quantization
2	3	9	13	12
3	8	15	18	18
4	15	19	22	24

It may be mentioned that in each of the above simulations, the adaptive techniques also registered an SNR gain of 2 to 4 dB over optimized nonadaptive quantizers. Table IX shows some results pertaining to a band-pass-filtered speech sample. These results were obtained from an independent experiment on coder assessment.<sup>12</sup> The adaptive quantizers (APCM, ADPCM) used the multipliers of Table VIII\*, and the nonuniform quantizer characteristics employed in adaptive DPCM are those recommended by Paez and Glisson.<sup>13</sup> Finally, the log-PCM used a  $\mu = 100$ ,<sup>7</sup> and the adaptive quantizers used a maximum-to-minimum-step-size ratio of 100.

Notice from the table that adaptive quantization, as incorporated into PCM, has the potential of outperforming the conventional technique of logarithmic companding. Evidently the advantages over log-PCM are even more impressive in ADPCM, and a companion paper will discuss, at length, the use of 3-bit and 4-bit adaptive quantizers in the DPCM coding of speech.<sup>1</sup>

#### V. ACKNOWLEDGMENTS

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\* Strictly speaking, these values are suboptimal for the nonuniform quantization in the last column of Table IX.

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